1: Four squares, all of side length 1, are arranged into a single figure such that each square is connected by an edge. What is the difference between the minimum and maximum possible perimeters of such figures?

Solution:
2. Three connections are required to bring together the 4 components into one figure, which means the theoretical maximum is achieved by $16 - 6 = 10$, or arranging the 4 squares in a line. The minimum must utilize as many redundant connections as possible. WLOG, we can connect two squares. The third square can be connected in one of two ways: the line, or the L-shape. The line offers no possibility for a redundant connection for the fourth square, meaning the the minimum is achieved by arranging the 4 squares in a square, with a perimeter of 8.

2: How many integers satisfy the inequality $|-2|x|+3| < 15$

Solution:
17.
Transforming the absolute value:

$$\begin{gathered}-15 < -2|x| + 3 < 15 \\ -18 < -2|x| < 12 \\ -9 < -|x| < 6 \\ 9 > |x| > -6 \\ 9 > |x| \geq 0 \\ -9 < x < 9\end{gathered}$$

Yielding 17 possible solutions.

3: If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{x+2}{x}$, evaluate $f(g(2))$

Solution:
$\frac{1}{4}$.

4: Find the sum of the real solutions to $x^2 + 12x + 35 = 0$.

Solution:
-12. Note that if the solutions are r_1, r_2, then $(x - r_1)(x - r_2) = x^2 + 12x + 35$, which means that $12 = -r_1 - r_2$, i.e. $r_1 + r_2 = -12$.

5: Solve for x: $log_2(x^2) + log_2(2x) = 4$

Solution:
2.
Using log properties, we have $2log_2(x) + log_2(x) + log_2(2) = 4$.
Then, $3log_2(x) = 3 \implies log_2(x) = 1 \implies x = 2$

6: Jenny is twice as old as Craig, and Craig is 5 years older than Lucy.

If Jenny is 3 times older than Lucy, how old is she?

Solution:
30.
Let Jenny's age be J, Craig's C, and Lucy's L. Then $C = L + 5$, $J = 2C$, and $J = 3L$. Solving, we have $J = 2L + 10 = 3L \implies L = 10$ and $J = 30$.

7: Let f be a quadratic equation $ax^2 + bx + c$ such that $f(0) = f(1), f(2) = 2f(1)$, and $f(3) = 16$. Calculate $f(-1)$.

Solution:
8
We have the following information:
$a + b + c = c$
$4a + 2b + c = 2a + 2b + 2c$
$9a + 3b + c = 16$
From the first equation, we have $a = -b$.
From the second, we have $2a = c$
Substituting into the last equation, we have $9a - 3a + 2a = 16 \implies a = 2$, from which we get $b = -2, c = 4$.
Thus $f(-1) = 8$.

8: Farmer Arnold raises chickens, but wants to eat bacon. Farmer Schwartz raises pigs, but wants to eat fried chicken. They want to trade, but they are also sticklers for being fair. If they agree that one chicken is worth 7 dollars, and one pig is worth 11 dollars, what is the smallest dollar value of a fair trade?

Solution:
77.
Since $7, 11$ are relatively prime, the smallest fair trade possible is $7 * 11$.

9: Prices have increased and chickens are now 14 dollars and a pig is 21 dollars. However, Farmer Schwartz realizes it's a pain to move pigs one at a time, so he will only trade pigs in sets of 3. What is the smallest dollar value of a fair trade?

Solution:
126.
The pigs in sets of 3 condition essentially makes Farmer Schwartz's trade unit size 63, and the lcm of $14, 63 = 126$.

10: Given a circle with equation $x^2 + y^2 = 25$ and an ellipse with equation $(\frac{x}{3})^2 + (\frac{y}{4})^2 = 1$, how many intersection points are there?

Solution:
0.
The ellipse has minor axis length 3 and major axis length 4, which means it lies

entirely within the circle of radius 5.

11: Let a, b, c be the solutions to $x^3 - 8x^3 + 10x + 10$. Find $a + b + c$.

Solution:
8.
Since $(x-a)(x-b)(x-c) = x^3 - 8x^2 + 10x + 10$, we have $-8 = -a - b - c$.

12: Given a non-degenerate triangle with integer side lengths, such that two legs have lengths 14 and 11, what is the longest possible third side length?

Solution:
24.
By the triangle inequality, if x is the third leg length, $x < 11 + 14$, then 24 is the largest such integer.

13: A shopper was buying scarves and sweaters, 13 in total. If scarves cost 2 dollars less than sweaters, both prices are integers, and the shopper spent 90 dollars altogether, how much do sweaters cost?

Solution:
8.
If x is the number of sweaters and y is the cost of a sweater, then $xy + (y - 2)(13 - x) = 90$, so $xy - xy + 13y + 2x - 26 = 90$, so $13y + 2x = 116$. Conditioning that $0 \leq x \leq 13$, we see that $y = 8, x = 6$.

14: Jon, Rick, and Amy share a lawn. If Jon and Rick can mow the lawn in 4 hours, Rick and Amy can mow it in 5 hours, and Jon and Amy can mow it in 4.5 hours, how quickly can the three of them together mow the lawn?

Solution:
$\frac{360}{121}$ hours.
Denote Jon J, Rick R, and Amy A.
JR can mow $\frac{1}{4}$ of a lawn per hour, RA can mow $\frac{1}{5}$ of a lawn per hour, and JA can mow $\frac{2}{9}$ of a lawn per hour.
That means $J + R = \frac{1}{4}$, $R + A = \frac{1}{5}$, and $J + A = \frac{2}{9}$. Solving for J, we have $J = \frac{1}{20} + A \implies 2A = \frac{2}{9} - \frac{1}{20} \implies A = \frac{1}{9} - \frac{1}{40} = \frac{31}{360}$.
This yields $J = \frac{49}{360}$ and $R = \frac{41}{360}$, so summing all 3 rates up, we find that combined they can mow $\frac{121}{360}$ of a lawn per hour, yielding $\frac{360}{121}$ hours to mow the entire thing.

15: Which of the following statements is false?

Solution:
- a) The sum of two even functions is even.
- b) The sum of two odd functions is odd.
- c) The product of two even functions is even.

d) The product of two odd functions is odd.
d) is false

16: Solve for x: $\sqrt{x+\sqrt{7+\sqrt{x+\sqrt{7+...}}}}=2$

Solution:
5.
Substituting the equation back into itself, we get the following:
$\sqrt{x+\sqrt{7+2}}=2 \implies x+3=4 \implies x=1$

17: Which of the following polynomials has no rational roots?

Solution:
a) x^2+4x+4
b) $x^3+7x^2+14x+7$
c) x^4+6x^2-40
d) $x^3+3x^2-4x-12$
By Eisenstein's criterion, b has no rational roots.

18: If $sec(x)+tan(x)=\frac{7}{3}$, what is $sec(x)-tan(x)$?

Solution:
From the property $sec^2(x)=tan^2(x)+1$, we know that $sec(x)-tan(x)=\frac{3}{7}$

19: Given $sin(2x+y)=\frac{2}{5}$ and $sin(2x-y)=\frac{3}{5}$, what is the value of $sin(x)cos(x)cos(y)$?

Solution:
$\frac{1}{4}$
Using the sum of sines formulas, we have $sin(2x)cos(y)+sin(y)cos(2x)=\frac{2}{5}$ and $sin(2x)cos(y)-sin(y)cos(2x)=\frac{3}{5}$, which yields $2sin(2x)cos(y)=1 \implies sin(2x)cos(y)=\frac{1}{2} \implies 2sin(x)cos(x)cos(y)=\frac{1}{2} \implies sin(x)cos(x)cos(y)=\frac{1}{4}$.

20: If I throw a dart at a circle of radius 1 centered at the origin, which of the following inequalities represents a region which the dart has a $1-\frac{2}{\pi}$ probability of landing in?

Solution:
a) $x+y\geq 0$
b) $x\leq\frac{1}{2}$
c) $||x|+|y||>1$
d) $x+2y>-.2$
We quickly note that a,d cover at least half of the circle, and thus are impossible. The area of region c can be computed as $4*(\frac{\pi}{4}-\frac{1}{2})=\pi-2$, precisely the area needed.

21: Ron paddles his canoe upstream from his house to the park and gets there in 2 hours. On the way back, the trip takes 1 hour. Assuming the park is 4 miles away and Ron is paddling at a constant rate, what is the speed of the current?

Solution:
1 mph.
Let Ron's rate be r and the current be c. We know that $r - c = 2$ and $r + c = 4$, hence $c = 1$.

22: This time, Ron takes a different river and goes to the zoo. The trip again takes 2 hours upstream and 1 our downstream, but he forgot to measure how far the zoo was. If he knows the zoo was at most 5 miles away, what is the maximum speed of the current, given that he was traveling at a constant rate?

Solution:
$\frac{5}{4}$.
With the same variables defined above, we add the distance $d \leq 5$. We know that $r - c = \frac{d}{2}$ and $r + c = d$, then the value of $c = \frac{d}{4}$ and thus, is maximized when $d = 5$, i.e., $c = \frac{5}{4}$.

23: John, Jack, Jumpy, and Jot are out to watch a movie, and you aree standing in line behind them. John pays for the tickets and then notices that the tickets were not the same price. Jot asks, "Well, how much were they?" You lean in to listen to what John says, "The cost of our tickets multiply to 4608 (which factorizes as $2^9 * 3^2$) and sum to ...", but you can't quite catch the last bit. Jot says, "Well, that's not enough information...", and John says, "Yours was the most expensive." You butt in and say, "Jot, your tickets cost..."

Solution:
a) 6,6,8,16
b) 8,8,8,9
c) 4,8,9,16
d) 3,8,8,24

6,6,8,16 dollars.
Let the 4 prices be a, b, c, d. We know that $abcd = 4608$. Moreover, we know that $abcd = 4608, a + b + c + d = x$ is not sufficient information to characterize a, b, c, d, but the fact that Jot's is the most expensive then yields enough information to solve. That means that we are looking for a sum of ticket costs such that multiple factorizations share that sum, but the existence of a most expensive ticket distinguishes that factorization. $6, 6, 8, 16$ is such a factorization, with $4, 8, 12, 12$ being its counterpart (both sum to 36). The final distinguishing component is that Jot's is the most expensive ticket, meaning that $4, 8, 12, 12$ cannot be the factorization.

24: Assuming x, y, z are positive real numbers, given $x^3 + y^3 + z^3 = 4$, what is the maximum value of $(1 + x^3)(1 + y^3)(1 + z^3) - (yz)^3 - (xz)^3 - (xy)^3$?

Solution:
$\frac{199}{27}$.
Expanding the final expression, we get $1 + x^3 + y^3 + z^3 + (xyz)^3 = 5 + (xyz)^3$.
By AM-GM, we know that $\frac{x^3+y^3+z^3}{3} \geq \sqrt[3]{x^3y^3z^3} \implies xyz \leq \frac{4}{3}$
Thus, the entire expression is less than or equal to $5 + \left(\frac{4}{3}\right)^3 = \frac{199}{27}$.

25: Given $f(x) - xf(-x) = 5$, compute $f(1)$.

Solution:
4.
Substituting $x = 1$, we see that $f(1) - f(-1) = 5$. Taking advantage of symmetry and plugging in $x = -1$, we note that $f(-1) + f(1) = 5$. Summing the two equations, we quickly see that $f(1) = 5$.

26: Given $f(n) = f(n + 1)f(n - 1)$, and $f(1) = 1, f(2) = 2$, compute $\sum_{i=1}^{2013} log_2(f(i))$.

Solution:
Rewriting the givens, we note that $f(n) = \frac{f(n-1)}{f(n-2)}$. Moreover, we note that by utilizing log properties, the sum telescopes so that we are left with $log_2(f(2013)) + log_2(f(2))$. We note that the sequence $f(i)$ is periodic with period 6, such that $f(1) = 1, f(2) = 2, f(3) = 2, f(4) = 1, f(5) = \frac{1}{2}, f(6) = \frac{1}{2}, ...,$ which means $f(2013) = f(3) = 2$, so the answer is $log_2(2) + log_2(2) = 2$.

27: Given $xy = x + y, yz = y + z, xz = x + z$, compute $\frac{xyz}{xy+yz+xz}$.

Solution:
$\frac{2}{3}$.
Dividing the every equation by the LHS, we have that $\frac{1}{x}+\frac{1}{y} = \frac{1}{y}+\frac{1}{z} = \frac{1}{x}+\frac{1}{z} = 1$, and so summing, we have that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2} = \frac{xy+yz+xz}{xyz} \implies \frac{xyz}{xy+yz+xz} = \frac{2}{3}$.

28: Let f be a function such that if $x \in \text{Domain}(f)$, then $\frac{1}{x} \in \text{Domain}(f)$. If $f(x)f(\frac{3}{x}) = x$, solve for the product of all values x that satisfy the above equation.

Solution:
-3.
Begin by substituting $\frac{3}{x}$ in for x, yielding the functional equation $f(\frac{3}{x})f(x) = \frac{3}{x}$. Since $f(x)f(\frac{3}{x}) = f(\frac{3}{x})f(x)$, we have $\frac{3}{x} = x \implies x^2 = 3 \implies x = \pm\sqrt{3}$, mean-

ing the product of the solutions is $-\sqrt{3}\sqrt{3} = -3$.

29: How many integer solutions (m, n) exist to the following equation: $n^2 + 4n + 2 = (m+1)(m+2)(m+3)(m+4)$?

Solution:
0.
We note that $n^2 \equiv 0, 1 \mod 4$, meaning $n^2 + 4n + 2 \equiv 2, 3 \mod 4$.
One of $m+1, m+2, m+3, m+4$ is a multiple of 4, however, which means that $(m+1)(m+2)(m+3)(m+4) \equiv 0 \mod 4$.

This means there are no possible integer solutions.

30: Given a, b, c are positive reals, what is the minimum value of $a(1+a^2) + b(1+b^2) + c(1+c^2)$ given $abc = 6$?

Solution:
$18 + \sqrt[3]{6}$

Expanding out the given, we have $a^3 + b^3 + c^3 + a + b + c$.

With two applications of AM-GM, we see that $a^3 + b^3 + c^3 \geq 3abc$, and that $a + b + c \geq 3\sqrt[3]{abc}$, which yields a minimum of $18 + 3\sqrt[3]{6}$. The only thing left to be verified is that equality is attained at the same point, which one can see with $a = b = c = \sqrt[3]{6}$.