Note: For all questions, answer "(E) NOTA" means none of the above answers is correct. For all questions, unless otherwise noted, assume that $i=\sqrt{-1}$. You may also find it useful to know that $2013=3\cdot 11\cdot 61$.

1. Which of the following best describes the series $\sum_{k=1}^{\infty} a_k$ where $a_k = 1/k$?

   (A) Fractional (B) Harmonic (C) Inverse (D) Reciprocal (E) NOTA

2. In the previous question, we had the general term of the sequence as $a_k = \frac{1}{k}$. This is very important, because without specifying this term, the value of a series could be ambiguous. For example, if asked to evaluate $\frac{1}{2}+\frac{1}{2}+\frac{3}{8}+\frac{1}{4}+\cdots$, one might interpret this as $S_1=\sum_{k=1}^{\infty} m_k$ where $m_k=\frac{k}{2^k}$, or as $S_2=\sum_{k=1}^{\infty} n_k$ where $n_k=\frac{1}{2}\left(\frac{3}{4}\right)^{k-1}+\left(\frac{1}{2}\right)^k$. Find S_2.

   (A) $\frac{5}{2}$ (B) 3 (C) $\frac{7}{2}$ (D) 5 (E) NOTA

For Questions 3-5, consider the sequence $a_k=\cos\left(\frac{k\pi}{3}\right)+i\sin\left(\frac{k\pi}{3}\right)$ for integers $k\geq 1$.

3. Find a_{2013}.

   (A) -1 (B) $-i$ (C) i (D) 1 (E) NOTA

4. Compute $\sum_{k=1}^{2013} a_k$.

   (A) $-1-i\sqrt{3}$ (B) $-1+i\sqrt{3}$ (C) $1-i\sqrt{3}$ (D) $1+i\sqrt{3}$ (E) NOTA

5. Compute $\prod_{k=1}^{2013} a_k$.

   (A) -1 (B) $-i$ (C) i (D) 1 (E) NOTA

6. Consider the geometric sequence $a_1, a_2, a_3, \ldots$ where $a_1 = \sin(x)$ and $a_2 = -\sin^2(x)$. Given that $\sum_{k=1}^{\infty} a_k = \frac{1}{3}$, find a possible value of x on the interval $0 \le x \le \frac{\pi}{2}$.

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$ (E) NOTA

7. Consider the sequence $a_k = \sin\left(\frac{k\pi}{61}\right)$. If $D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$, find $\sum_{k=1}^{2013} D(a_k)$.

(A) 33 (B) 34 (C) 66 (D) 132 (E) NOTA

8. Define a sequence of functions $f_1(x)$, $f_2(x)$, $f_3(x)$, ... such that $f_1(x) = x$ and $f_{n+1}(x) = \frac{1}{1 - f_n(x)}$ for integers $n \ge 1$. Find the value of $f_{2013}(2013)$.

(A) -2012 (B) $-\frac{1}{2012}$ (C) $\frac{2012}{2013}$ (D) 2013 (E) NOTA

9. Consider the arithmetic sequence $a_1, a_2, a_3, \ldots$ where $a_1 = \sin\left(\frac{\pi}{12}\right)$ and $a_2 = \cos\left(\frac{\pi}{12}\right)$. Find the value of a_3.

(A) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{2}+\sqrt{6}}{4}$ (D) $\frac{3\sqrt{2}+\sqrt{6}}{4}$ (E) NOTA

For Questions 10-11, consider the sequence $a_k = \binom{k+3}{k}$ for integers $k \ge 0$.

10. Which of the following best describes the sequence?

(A) arithmetic (B) geometric (C) monotonic (D) periodic (E) NOTA

11. Find the remainder when $\sum_{k=0}^{2013} a_k$ is divided by 2013.

(A) 0 (B) 1 (C) 3 (D) 672 (E) NOTA

12. Find the sum of all possible values for the first term of a geometric sequence with second term $1+i$ and fifth term $2+2i$.

(A) 0 (B) $\frac{2(1+i)}{3}$ (C) $\frac{\sqrt[3]{4}\cdot(1+i)}{2}$ (D) 2 (E) NOTA

13. Consider the real-valued sequences a_k and b_k such that $a_k + ib_k = \left(1+i\sqrt{3}\right)^k$ for all positive integers k. Compute $a_{20}b_{13} + b_{20}a_{13}$.

(A) -2^{33} (B) 2^{32} (C) $2^{32}\sqrt{3}$ (D) 2^{33} (E) NOTA

14. Consider the sequence 1, 1, 1, 2, 2, 1, 1, 2, 3, 3, 2, 1, 1, 2, 3, 4, 4, 3, 2, 1, 1, 2, 3, 4, 5, 5, 4, 3, 2, 1,…, where the terms increase from 1 through n, inclusive, and then decrease from n to 1, inclusive, for n = 1, 2, 3, … What is the 2013th term in this sequence?

(A) 30 (B) 31 (C) 32 (D) 33 (E) NOTA

15. Compute $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{3^k}$.

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) 1 (E) NOTA

16. Compute $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{3^k}$.

(A) $\frac{13}{12}$ (B) $\frac{5}{4}$ (C) $\frac{3}{2}$ (D) $\frac{7}{4}$ (E) NOTA

17. Compute $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3}{3^k}$.

(A) $\frac{15}{4}$ (B) 4 (C) $\frac{65}{16}$ (D) $\frac{33}{8}$ (E) NOTA

For questions 18-20, consider the sequence defined by $a_{k+1} = \frac{a_k}{2} + \frac{1}{a_k}$ **for integers** $k \geq 1$.

18. If $a_1 = 1$, find a_3.

(A) $\frac{17}{12}$ (B) $\frac{3}{2}$ (C) $\frac{13}{6}$ (D) $\frac{5}{2}$ (E) NOTA

19. If $a_1 = 1$, find $\lim_{k \to \infty} a_k$.

(A) $\frac{\sqrt{6}}{2}$ (B) $\sqrt{2}$ (C) $\frac{3}{2}$ (D) $\frac{1+\sqrt{5}}{2}$ (E) NOTA

20. Interestingly, when any value of a_k is written as a fraction in simplest form, $\frac{m}{n}$, then $m^2 - 2n^2 = 1$ or $m^2 - 2n^2 = -1$. This is because each a_k is, in effect, an approximation for $\lim_{k \to \infty} a_k$. Using this idea as inspiration, you can find the positive integral value of M such that $\frac{M}{72}$ is as close as possible to $\sqrt{5}$. What is the sum of the digits of M?

(A) 8 (B) 9 (C) 10 (D) 11 (E) NOTA

21. The first term in an arithmetic sequence is 1 and the fourth term is 3. Find the second term in this sequence.

(A) 1.5 (B) $\frac{5}{3}$ (C) 2 (D) $\frac{7}{3}$ (E) NOTA

22. Here, we will consider a sequence of vectors; specifically, we consider the sequence $a_k = \left\langle \frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2} \right\rangle$ for integers $k \geq 1$. We then define the real-valued sequence $b_k = a_k \bullet a_{k+1}$ for integers $k \geq 1$. Find $\sum_{k=1}^{\infty} b_k$.

(A) $\frac{3}{8}$ (B) $\frac{3}{2}$ (C) $\frac{11}{6}$ (D) ∞ (E) NOTA

23. Consider the sequences $a_k = \cos(1+2013k)+\cos(3+2013k)$ and $b_k = \cos(2+2013k)$ for integers $k \geq 0$. Find the value of $\lim_{n\to\infty}\left[\frac{\sum_{k=0}^{n} a_k}{\sum_{k=0}^{n} b_k}\right]$. *Hint:* Use an identity for $\cos(\alpha)+\cos(\beta)$.

(A) $2\cos(2)$ (B) $\cos(2)$ (C) $\cos(1)$ (D) $2\cos(1)$ (E) NOTA

24. Evaluate $\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)}{k!}\right]$.

(A) $e-2$ (B) $e-1$ (C) e (D) $e+1$ (E) NOTA

For questions 25-27, consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for integers $k \geq 1$.

25. Evaluate $\sum_{k=1}^{\infty} a_k$.

(A) $\frac{4}{5}$ (B) $\frac{9}{10}$ (C) $\frac{19}{20}$ (D) 1 (E) NOTA

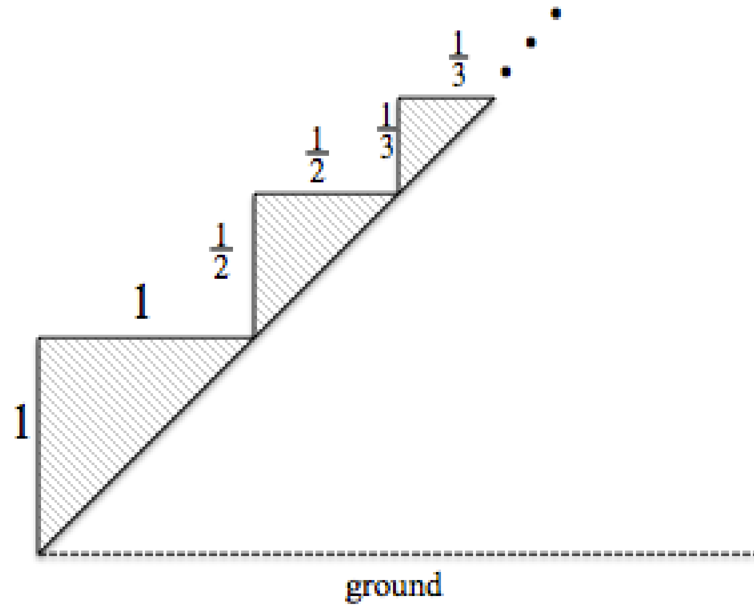
26. If $\sum_{k=m}^{n-1} a_k = \frac{1}{2013}$ where m and n are positive integers with $m < n$, find m in terms of n.

(A) $\frac{n+2013}{2013n}$ (B) $\frac{2013n}{2013-n}$ (C) $\frac{2013n}{n-2013}$ (D) $\frac{2013n}{n+2013}$ (E) NOTA

27. If $\prod_{k=1}^{2013} a_k = \frac{1}{T}$, compute $\left\lfloor \frac{\sqrt{T}}{2013!} \right\rfloor$. (Note: $\lfloor x \rfloor$ denotes the greatest integer function.)

(A) 0 (B) 44 (C) 45 (D) 2013 (E) NOTA

For Questions 28-30, consider the following staircase that Eli is climbing,



where the height and width of the nth stair is $\frac{1}{n}$. Let the sequence h_k denote the vertical distance from Eli's feet to the ground when he has walked onto the kth stair.

28. Which of the following intervals contains h_{2013}?

(A) $[1,5]$ (B) $[5,15]$ (C) $[15,100]$ (D) $[100,1000]$ (E) NOTA

29. Which of the following intervals contains $\lim_{k\to\infty} h_k$?

(A) $[1,10]$ (B) $[10,100]$ (C) $[100,1000]$ (D) $[1000,10000]$ (E) NOTA

30. Let the sequence a_k denote the total area of the sides of the stairs (the shaded region) that Eli has climbed when he reaches the kth stair. If all of the sides of the stairs are right triangles, compute $\lim_{k\to\infty} a_k$.

(A) $\frac{\pi^2}{12}$ (B) $\frac{\pi^2}{9}$ (C) $\frac{\pi^2}{6}$ (D) ∞ (E) NOTA