ANSWERS: BBABA AACDC DAEDC CDABA BCDBD DEBEA

1. **B** This is well known as the harmonic series.

2. **B** $\sum_{k=1}^{\infty}\left(\frac{1}{2}\left(\frac{3}{4}\right)^{k-1}+\left(\frac{1}{2}\right)^{k}\right)=\frac{1}{2}\cdot\frac{1}{1-\frac{3}{4}}+\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{1}{2}\cdot 4+1=3.$

3. **A** $\cos\left(\frac{2013\pi}{3}\right)+i\sin\left(\frac{2013\pi}{3}\right)=\cos(\pi)+i\sin(\pi)=-1.$

4. **B** This cancels out each time around the unit circle. Since we start with *k=1*, our desired value is $\sum_{k=2011}^{2013}a_k=cis\left(\frac{\pi}{3}\right)+cis\left(\frac{2\pi}{3}\right)+cis(\pi)=-1+i\sqrt{3}.$

5. **A** Note that $\prod_{k=1}^{2013}a_k=a_{\sum_{k=1}^{2013}k}$, and $\sum_{k=1}^{2013}k=\frac{2013\cdot 2014}{2}$, so $cis\left(\frac{2013\cdot 2014\pi}{2\cdot 3}\right)=cis(odd\cdot\pi)=-1.$

6. **A** The sum is $\frac{\sin(x)}{1+\sin(x)}=\frac{1}{3}\Rightarrow 3\sin(x)=1+\sin(x)\Rightarrow\sin(x)=\frac{1}{2}$. So, $x=\frac{\pi}{6}$.

7. **A** Because 61 is prime, we can't get rational non-integer multiples of *pi* that we'd like (such as pi/2 or pi/6), so we only care for *k* such that 61 divides *k*. Since 2013/61 = 33, there are exactly 33 such *k*.

8. **C** It is easy to see that *f* is cyclic with period 3. Specifically, $f_1(x)=x,\ f_2(x)=\frac{1}{1-x}$, $f_3(x)=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x},\ f_4(x)=\frac{1}{1-\frac{x-1}{x}}=\frac{x}{1}=f_1(x)$. Since 2013 is divisible by 3, we have $f_{2013}(x)=f_3(x)=\frac{x-1}{x}$, so the desired answer is $\frac{2012}{2013}$.

9. **D** $a_3=a_2+(a_2-a_1)=2\cos\left(\frac{\pi}{12}\right)-\sin\left(\frac{\pi}{12}\right)=\frac{3\sqrt{2}+\sqrt{6}}{4}.$

10. **C** It is increasing in *k*, so it is monotonic.

11. **D** Note that $\binom{k+3}{k}=\binom{k+3}{3}$, so by the Hockey-Stick identity,

$\sum_{k=0}^{2013}\binom{k+3}{3}=\sum_{k=3}^{2016}\binom{k}{3}=\binom{2017}{4}=\frac{2017\cdot 2016\cdot 2015\cdot 2014}{24}=84\cdot 2017\cdot 2015\cdot 2014.$ Taking this mod 2013 gives $84\cdot 4\cdot 2\cdot 1=672.$

12. **A** We have $r^3=2,$ and the desired sum is $(1+i)\sum\frac{1}{r}.$ However, the sum of the reciprocals of the solutions for r is 0 since there is no r term in the polynomial. Hence, the sum is 0.

13. **E** First, write $a_k+ib_k=2^k\cdot cis\left(\pi/3\right),$ so $a_k=2^k\cos\left(\frac{\pi k}{3}\right), b_k=2^k\sin\left(\frac{\pi k}{3}\right).$ By the sine addition formula, $a_{20}b_{13}+b_{20}a_{13}=2^{33}\sin\left(\frac{20\pi}{3}+\frac{13\pi}{3}\right)=2^{33}\sin(11\pi)=0.$

14. **D** The part of the sequence that "peaks" at n has a total of $2n$ terms. Thus, the sub-sequences peaking at $n=1,2,\ldots,k$ are a total of $2(1+2+\ldots+k)=k(k+1)$ terms. Note that 44(45) = 1980, so the 1980th term completes the 44 peaking sub-sequence. Then, 1981 is 1, 1982 is 2, …, 2013th is 33.

15. **C** (See "beta" in solution for 17)

16. **C** (See "alpha" in solution for 17)

17. **D**

Let

$$S = \sum_{x=1}^{\infty} \frac{x^3}{3^x} = \frac{1^3}{3^1} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \ldots$$

Then

$$3S = \sum_{x=1}^{\infty} \frac{x^3}{3^{x-1}} = \frac{1^3}{3^0} + \frac{2^3}{3^1} + \frac{3^3}{3^2} + \frac{4^3}{3^3} + \ldots$$

Subtracting to get $2S = 3S - S$ (by grouping terms with same denom.) gives

$$2S = \frac{1^3 - 0^3}{3^0} + \frac{2^3 - 1^3}{3^1} + \frac{3^3 - 2^3}{3^2} + \frac{4^3 - 3^3}{3^3} + \ldots$$

Consider that $(n+1)^3 - n^3 = 3n^2 + 3n + 1$, so this sum is

$$2S = \frac{3 \cdot 0^2 + 3 \cdot 0 + 1}{3^0} + \frac{3 \cdot 1^2 + 3 \cdot 1 + 1}{3^1} + \frac{3 \cdot 2^2 + 3 \cdot 2 + 1}{3^2} + \frac{3 \cdot 3^2 + 3 \cdot 3 + 1}{3^3} + \ldots$$

Grouping up n^2s, ns, and $1s$, this is

$$2S = 3\left(\frac{0^2}{3^0} + \frac{1^2}{3^1} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \ldots\right) + 3\left(\frac{0}{3^0} + \frac{1}{3^1} + \frac{2}{3^2} + \frac{3}{3^3} + \ldots\right) + \left(\frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\right)$$

Let's consider these separately. Let

$$\alpha = \frac{0^2}{3^0} + \frac{1^2}{3^1} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \ldots$$

Then

$$3\alpha = \frac{1^2}{3^0} + \frac{2^2}{3^1} + \frac{3^2}{3^2} + \ldots$$

so subtracting gives

$$2\alpha = 3\alpha - \alpha = \frac{1^2 - 0^2}{3^0} + \frac{2^2 - 1^2}{3^1} + \frac{3^2 - 2^2}{3^2} + \ldots = \frac{1}{3^0} + \frac{3}{3^1} + \frac{5}{3^2} + \frac{7}{3^3} + \ldots$$

Then,

$$6\alpha = 3(2\alpha) = 3 + \frac{3}{3^0} + \frac{5}{3^1} + \frac{7}{3^2} + \ldots$$

so subtracting gives

$$4\alpha = 6\alpha - 2\alpha = 3 + \frac{2}{3^0} + \frac{2}{3^1} + \frac{2}{3^2} + \ldots = 3 + \frac{2}{1 - \frac{1}{3}} = 6$$

Hence,

$$\alpha = \frac{6}{4} = \frac{3}{2}$$

Now, let

$$\beta = \frac{0}{3^0} + \frac{1}{3^1} + \frac{2}{3^2} + \frac{3}{3^3} + \dots$$

Then,

$$3\beta = \frac{1}{3^0} + \frac{2}{3^1} + \frac{3}{3^2} + \frac{4}{3^3} + \dots$$

so subtracting gives

$$2\beta = 3\beta - \beta = \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \dots = \frac{3}{2}$$

Hence,

$$\beta = \frac{3}{4}$$

We have that

$$2S = 3\alpha + 3\beta + 2\beta = 3\alpha + 5\beta = 3 \cdot \frac{3}{2} + 5 \cdot \frac{3}{4} = \frac{33}{4}$$

so

$$S = \frac{33}{8}$$

18. **A** $a_2 = \frac{1}{2} + \frac{1}{1} = \frac{3}{2} \Rightarrow a_3 = \frac{3/2}{2} + \frac{1}{3/2} = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}.$

19. **B** $L = \frac{L}{2} + \frac{1}{L} \Rightarrow \frac{L}{2} = \frac{1}{L} \Rightarrow L^2 = 2.$ Clearly, $L > 0$, so $L = \sqrt{2}.$

20. **A** Analogously, To get close to $\sqrt{5}$, we want $M^2 - 5 \cdot 72^2 = \pm 1.$ Through inspection, we find that $\sqrt{5 \cdot 72^2 + 1} = 161$, so the sum of the digits is 8.

21. **B** The common difference is (4-1)/3 = 2/3, so the answer is 1+2/3 = 5/3.

22. **C** $b_k = \frac{1}{k} \cdot \frac{1}{k+1} + \frac{1}{k+1} \cdot \frac{1}{k+2} + \frac{1}{k+2} \cdot \frac{1}{k+3}.$ The sum is then very easy to telescope. We see that $\sum_{k=a}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{a}$, so the desired value is $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}.$

23. **D** Using the hint: $a_k = \cos(1 + 2013k) + \cos(3 + 2013k) = 2\cos(2 + 2013k)\cos(1).$ Then, the desired value becomes $2\cos(1) \cdot \frac{\sum b_k}{\sum b_k} = 2\cos(1).$

24. **B** $\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)=\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-(k-1)}{n}\right)$, so $\frac{\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)}{k!}=\frac{1}{n^k}\cdot\binom{n}{k}$. Considering the binomial expansion of $\left(1+\frac{1}{n}\right)^n$ and noting that the *k=0* term is missing, we can say that our limit is $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n-1\right]=e-1$.

25. **D** $\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots=1$.

26. **D** $\sum_{k=m}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)=\frac{1}{m}-\frac{1}{n}$. Setting this equal to $\frac{1}{2013}$ gives $\frac{1}{m}=\frac{1}{n}+\frac{1}{2013}\Rightarrow\frac{1}{m}=\frac{2013+n}{2013n}\Rightarrow m=\frac{2013n}{2013+n}$

27. **B** Quite simply, $\prod_{k=1}^{2013}\frac{1}{k(k+1)}=\prod_{k=1}^{2013}\frac{1}{k}\cdot\prod_{k=1}^{2013}\frac{1}{(k+1)}=\frac{1}{2013!\cdot 2014!}\Rightarrow\sqrt{T}=2013!\sqrt{1+2014}$. The greatest integer less than or equal to $\sqrt{2015}$ is 44.

28. **B** There are many ways to approach this; use estimation techniques, noting that the terms at the beginning are very important, while later terms can be grouped into sets of 100 or even 500. The true value is about 8.48.

29. **E** It is well known that the harmonic series diverges.

30. **A** $\sum_{k=1}^{\infty}\left(\frac{1}{2}\cdot\frac{1}{k}\cdot\frac{1}{k}\right)=\frac{1}{2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{2}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{12}$.