Alpha Trigonometry

Problem	Question	Answer	Solution
1	Evaluate: $\cot\frac{\pi}{6}$	C - $\sqrt{3}$	By definition, $\cot x = \frac{\cos x}{\sin x}$. Evaluating this for $x = \pi/6$, we obtain that $\cos x = \sqrt{3}/2$ and $\sin x = 1/2$. Thus, $\cot x = \sqrt{3}$.
2	How many petals does the polar graph of $r = 43\cos(2013\theta)$ have?	C- 2013	The number of petals on this graph is determined by the coefficient of theta. Since the coefficient is odd, the number of petals is just equal to the coefficient, 2013.
3	Which of the following angles in radians is equivalent to 36 arcseconds?	B - $\frac{\pi}{18000}$	There are 60 arcminutes in a degree and 60 arcseconds in an arcminute. Recall that there are $\frac{\pi}{180}$ radians to a degree, thus 36 arcseconds is equivalent to $\frac{\pi}{18000}$.
4	Express the following in terms of cosine functions: $\sin(\arctan(\cos x))$.	D - $\frac{\cos x}{\sqrt{1+\cos^2 x}}$	A useful diagram is a right triangle with angle x, whose opposite leg is of length $\cos x$ and adjacent leg of length 1 and the hypotenuse of length $\sqrt{1+\cos^2 x}$. From this triangle we find that $\sin x = \frac{\cos x}{\sqrt{1+\cos^2 x}}$.
5	Which of the following trigonometric functions is odd?	B - $\sin 2x$	A function $f(x)$ is odd if the following property holds: $f(-x) = -f(x)$.
6	Two rays form an angle in the Cartesian plane. If one of the ray	A - $-\frac{5}{13}$	Use the trigonometric identity: $\tan^2 x + 1 = \sec^2 x$ to solve for the cosine. We take the negative solution as the angle is in

	is the positive x-axis, the other ray points towards Quadrant III, and the tangent of the resulting angle is $\frac{12}{5}$, what is the cosine of this angle?		the third Quadrant.
7	What is the sum of the solutions of the equation $\tan\theta=\sqrt{3}$, in the domain $(-2\pi,2\pi)$?	D - $-\frac{2\pi}{3}$	The solutions to the equation are of the form $\theta=\frac{\pi}{3}+k\pi$ for integer values of k. Summing the solutions in the specified domain yields D.
8	The sides of a triangle measure 65, 72, and 97. This triangle is:	C - Right	The sides of this triangle satisfy the Pythagorean identity. Hence the sides form a right triangle.
9	Restricted to $[0,2\pi)$, what is the argument of $3+i\sqrt{3}$? Note: $i=\sqrt{-1}$.	C - $\frac{\pi}{6}$	Rewriting the complex number in polar form, we find that the argument (or polar angle) is C.
10	What is the cosine of the angle between the complex vectors $1+2i$ and $2-3i$?	A - $-\frac{4\sqrt{65}}{65}$	The cosine of the angle between any two complex vectors is defined as: $\cos\theta=\frac{\text{Re}(a\cdot b)}{\lVert a\rVert\lVert b\rVert}$.
11	A helix is parameterized by $x=4\cos t$, $y=3\sin t$, and $z=t$. What is the distance between the points at time $t=0$ and $t=\pi$?	C - $\sqrt{64+\pi^2}$	The points in question are $(4,0,0)$ at time $t=0$ and $(-4,0,\pi)$ at time $t=\pi$.
12	Express the line $3x+5y=7$ in polar coordinates.	A - $r=\frac{7}{3\cos\theta+5\sin\theta}$	Use the change of variables: $x=r\cos\theta$ and $y=r\sin\theta$.

13	Simplify: $\sum_{n=0}^{\infty} 2\cos^n 2\theta$ for $0 \le \theta < \pi/2$.	D - $\csc^2 \theta$	Realize that the series is an infinite geometric one. Thus the sum , given the proper domain for convergence, converges to $\frac{2}{1-\cos 2\theta}$, which after using the double angle formula for cosine reduces to D.
14	What is the maximum value of the function: $y = \cos^4 x - \sin^2 x$?	B - $f(x) \le 1$	It may help to draw the two functions $y = \cos^4 x$ and $y = \sin^2 x$. It is clear that the maximum difference between these two occurs when one reaches a maximum value of one and the other a minimum value of 0.
15	What is the period of the graph of $y = \sin^5 x + \sin^2 x$?	C - 2π	First function has period pi. Second function has period 2pi. Sum yields a period of 2pi.
16	One of the first trigonometric functions is the chord function, defined as: $\text{crd}(\theta) = 2\sin\frac{\theta}{2}$. What is $\cos^2 \theta$ in terms of this ancient function?	D - $1 - 4\text{crd}(\theta) + 4\text{crd}^2(\theta)$	Using the half-angle formula for sine, we can solve for cosine in terms of the chord function.
17	What is the entry in the first row, first column of the matrix M^2, for $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$?	A - $1 - 2\sin^2 \theta$	By matrix multiplication, the entry is $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$.
18	The trigonometric ratio $\sec\frac{\pi}{12}$ can be expressed in the form of	C - -24	We begin with evaluating cosine at the angle $\pi/12$ using the half angle formula for cosines.

	$a\sqrt{\sqrt{b}+c\sqrt{d}}$ where a,b,c,d are integers and a even. What is the product $abcd$?		$\cos \pi/12 = \cos \frac{\pi/6}{2}$ $= \sqrt{\frac{1+\cos \pi/6}{2}}$ $= \frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}$ $= \frac{1}{2}\sqrt{2+\sqrt{3}}$ Thus, the secant of $\pi/12$ is $\sec\frac{\pi}{12} = \frac{1}{\cos \pi/12}$ $= \frac{2}{\sqrt{2+\sqrt{3}}}$ $= 2\sqrt{2-\sqrt{3}}$ Hence, $a=2 \quad c=-1$ $b=4 \quad d=3$ So the product is $abcd=-24$
19	What is the value of $\sum_{n=2}^{6} e^{\binom{n}{2}\pi i}$? Note: $i=\sqrt{-1}$.	D - -1	$\sum_{n=2}^{6} e^{\binom{n}{2}\pi i} = e^{\binom{2}{2}\pi i} + e^{\binom{3}{2}\pi i} + e^{\binom{4}{2}\pi i} + e^{\binom{5}{2}\pi i} + e^{\binom{6}{2}\pi i}$ $= -1 + -1 + 1 + 1 + -1$ $= -1$
20	What is the exact value of $\tan 75°$?	B - $2+\sqrt{3}$	Notice that the argument of the tangent function is a sum of 45° and 30°. We can either recall the angle sum formula for tangent or use the definition of tangent as a ratio of the sine and cosine functions and their respective angle sum

			formulas. Both are shown below: $\tan 75° = \sin 75° / \cos 75°$ $\tan(30°+45°) = \frac{\sin(30°+45°)}{\cos(30°+45°)}$ $\frac{\tan 30° + \tan 45°}{1-\tan 30° \tan 45°} = \frac{(\sin 30° \cos 45° + \cos 30° \sin 45°)}{(\cos 30° \cos 45° - \sin 30° \sin 45°)}$ $\therefore \tan 75° = 2+\sqrt{3}$
21	Triangle XYZ has side lengths of $x=5$, $y=6$, and $z=4$. What is the exact value of $\tan\frac{Y}{2}$?	A - $\frac{\sqrt{7}}{3}$	Using the law of cosines, we have that $\cos Y = \frac{1}{8}$. This will help when we consider the half angle formula for sine and cosine.
22	What is the range of $f(x)=\cot^{-1} x$?	E – NOTA	There are multiple possibilities for the range of arccotangent function and all are equally appropriate.
23	What is the probability that a randomly chosen angle $\theta \in [0, 2\pi)$ satisfies the inequality: $\cos 2\theta < \sin\theta$?	C - $\frac{1}{3}$	Using the double angle formula for cosine, we have that the inequality can be rewritten as $2\sin^2\theta + \sin\theta - 1 < 0$. This inequality holds for when $\sin\theta > \frac{1}{2}$, or $\theta \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$.
24	Solve for smallest positive value of x such that $\cos 3x + 3\cos x - 4 = 0$.	E	Multiples of 2π form the solution set for this equation.
25	How many distinct obtuse triangles can be made by selecting three distinct integers between 1 and 10, inclusive, as side lengths?	E	There are 15 acute triangles, 33 obtuse triangles, and 2 right triangles.
26	Given the triangle ABC, $a=4$,	C - $12-4\sqrt{3}$	Drawing an altitude, we have that the area of the triangle

	$m\angle A = 75°$, $m\angle B = 60°$, find the area of the triangle.		ABC is $2b\sin C$. The angle C can be determined using the property that the sum of the interior angle of any triangle is 180 degrees. The side length b can be determined by using the law of sines and the angle sum formula for sines.
27	Which of the following expressions has the smallest value?	A - $\cos 1$	Drawing a picture may help distinguish whether the sine function or cosine function is larger at 1 radian.
28	Which of the following expressions are equivalent to $\cos x + \sin x$ for all values of x?	A - $\sqrt{2}\cos\left(\frac{\pi}{4} + x\right)$	We begin by using the following definition for the cosine function: $\cos x = \sin\left(\pi/2 - x\right)$. Hence, $\cos x + \sin x = \sin\left(\pi/2 - x\right) + \sin x$ $= 2\sin\left(\pi/4\right)\sin\left(\pi/4 - x\right)$ $= \sqrt{2}\cos\left(\pi/4 + x\right)$ where the second equality comes from the summation identity: $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$, and the third from evaluating the sine function and applying the definition used earlier.
29	Solve for x: $\cos^{-1} x = \sin^{-1} 3x$.	A - $\frac{\sqrt{10}}{10}$	Take the cosine of both sides. The left side will reduce to x, while the right side can be calculated by considering the cosine of an angle whose sine ratio is 3x.
30	Which of the following polar coordinates are equivalent to the rectangular coordinates	B - $\left(4, \pi/5\right)$	From the solutions, we are inspired to consider multiples of $x = \frac{\pi}{5}$. Drawing a picture can help reveal that $\cos 2x = -\cos 3x$. Using the angle sum formula for cosines, we

	$\left(\sqrt{5}+1, \sqrt{10-2\sqrt{5}}\right)$?		find that $x = \frac{\pi}{5}$ will indeed yield rectangular coordinates that are a quarter of the length of the requested coordinate pair.