2015 Theta Equations and Inequalities
Answers

1. D
2. A
3. B
4. A
5. C
6. B
7. A
8. B
9. D
10. A
11. B
12. D
13. C
14. A
15. B
16. E Answer: $\left(\frac{18}{a-6}, \frac{-4(a+3)}{a-6}\right)$
17. A
18. A
19. B
20. D
21. E Answer: 2.5
22. C
23. C
24. A
25. B
26. D
27. C
28. D
29. C
30. D

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Solutions

1. To have no real roots, the discriminant must be negative. $16k^2-4(3k)(1)<0 \to 4k(4k-3)<0$.

   The solution interval is $\left(0,\frac{3}{4}\right)$, **D**.

2. Using Vieta's formula, the sum is found by $-\frac{(-8)}{4}=2$, **A**.

3. $\log_x y$ is equivalent to $\frac{1}{\log_y x}$. After multiplying the entire equation by $\log_y x$ and rearranging, we have $\left(\log_y x\right)^2-2.9\log_y x+1=0 \to 10\left(\log_y x\right)^2-29\log_y x+10=0$. This factors to $\left(5\log_y x-2\right)\left(2\log_y x-5\right)=0$, giving $\log_y x=\frac{2}{5}$ or $\frac{5}{2}$. These are equivalent, respectively, to $x=y^{\frac{2}{5}}$ or $y^{\frac{5}{2}}$. Substituting these into $xy=128$, we obtain $y=32, x=4$ and $y=4, x=32$. Either way, the sum is 36, **B**.

4. $2^x=\frac{1}{4}=2^{-2}$. $(-2)^{-2}=\frac{1}{4}$, **A**.

5. $3a^2-4ab+b^2=0 \to (3a-b)(a-b)=0$. This gives $\frac{a}{b}=\frac{1}{3}$ and $\frac{a}{b}=1$. Since a and b must be distinct,

   the ratio must be $\frac{1}{3}$, **C**.

6. $x=2$ is the only positive root of the equation. $x=-2$ is also a root, along with two imaginary roots.

7. $|x|^2-\sqrt{x^2}-6=0 \to |x|^2-|x|-6=0 \to \left(|x|-3\right)\left(|x|+2\right)=0$. The only real solutions come from $|x|=3 \to x=\pm 3$. These values are found in the interval in **A**.

8. Let x be the number of 5-cent increases. $R=(0.85+0.05x)(5000-200x)=10x^2-80x-4250$.

   This function has a maximum at $x=\frac{80}{2(10)}=4$. A 20-cent increase bring the total to $1.05, **B**.

9. $2x-\sqrt{2}<6 \to x<3+\frac{\sqrt{2}}{2}\approx 3.707$ and $1-2x<-4 \to x>\frac{5}{2}$. The answer is 3, **D**.

10. ${}_nP_4=20\left[{}_{n-1}C_2\right] \to \frac{n!}{(n-4)!}=\frac{20(n-1)!}{2!(n-1-2)!} \to \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!}=\frac{10(n-1)(n-2)(n-3)!}{(n-3)!}$.
    This simplifies to $n(n-3)=10 \to (n-5)(n+2)=0 \to n=5$, **A**.

11. Squaring each side, we get $x^2+4x+4=4+x\sqrt{8-x} \to x^2+4x=x\sqrt{8-x}$. Dividing by x, we lose a

solution of 0, which won't affect our final answer. Squaring again, we obtain $(x+4)^2 = 8 - x \rightarrow$ $x^2 + 9x + 8 = 0$. This factors to $(x+8)(x+1) = 0$, but the only true solution we get is $x = -1$, **B**.

12. The third side must be between 4 and 24 (exclusive), so the answer is **D**.

13. Using the quadratic formula, we get $x = \dfrac{-\sqrt{3} \pm \sqrt{3 - 4(-2)}}{2\sqrt{2}} \rightarrow \dfrac{-\sqrt{6} \pm \sqrt{22}}{4}$, **C**.

14. Using expansion of minors: $(x-4)(-3-8) - (x-2)(-1-8) + (-2)(2-6) = 28$. This simplifies to $x = 3$, **A**.

15. $\left(\log(x)\right)^3 = \log\left(x^{16}\right) \rightarrow \left(\log(x)\right)^3 = 16\log\left(x\right) \rightarrow c^3 = 16c \rightarrow c(c+4)(c-4) = 0 \rightarrow \log x = 0, 4, -4$. So, $x = 1$, $x = 10{,}000$, or $x = \dfrac{1}{10{,}000}$. Their product is 1, **B**.

16. Using substitution, $ax + 3(-2x - 4) = 6 \rightarrow x = \dfrac{18}{a-6}$. $y = -2\left(\dfrac{18}{a-6}\right) - 4 \rightarrow \dfrac{-4(a+3)}{a-6}$. **E**.

17. Using "stars and bars," $\dbinom{10+4-1}{10\ (or\ 3)} = 286$, **A**.

18. $x^2 - y^2 = 270 \rightarrow (x+y)(x-y) = 2 \cdot 3^3 \cdot 5$. $x + y$ and $x - y$ must be both even or both odd; however, 270 has only one factor of 2 so its factors are one even and one odd. Therefore there are no integral pair factors, **A**.

19. $XY \times 23 = 1XY1$, so $Y = 7$ since $3 \times 7 = 21$. $X7 \times 23 = 1X71$, so $(10X + 7) \times 23 = 1000 + 100X + 10(7) + 1$. This gives $X = 7$. $X + Y = 14$, **B**.

20. Rewrite the problem as $3^{4x^3 + 8x^2} = 3^{\frac{5x}{3}}$. Set the exponents equal to one another and multiply by 3 to clear the fraction. This gives $12x^3 + 24x^2 - 5x = 0$, which has a solution of 0, which does not affect our answer. Divide by *x* and then use the quadratic formula: $x = \dfrac{-24 \pm \sqrt{576 - 4(12)(-5)}}{24} = \dfrac{-6 \pm \sqrt{51}}{6}$. The sum of these solutions is –2, **D**.

21. Split the problem into $3 - 4x \geq 1$ *and* $3 - 4x \leq 11$. These result in $x \leq \dfrac{1}{2}$ *and* $x \geq -2$. The length of this interval is 2.5, **E**.

22. Let $u=\frac{1}{x}, v=\frac{1}{y}, w=\frac{1}{z}$. Now we have $\begin{cases} u+v=\frac{1}{3} \\ u+w=\frac{1}{5} \\ v+w=\frac{1}{7} \end{cases}$. Subtracting the first two and adding to the third gives us $v=\frac{29}{210}$. From the third equation, $w=\frac{1}{7}-\frac{29}{210}=\frac{1}{10}$. $\frac{z}{y}=\frac{\frac{1}{w}}{\frac{1}{v}}=\frac{v}{w}=29$, **C**.

23. Since *c* has to be a divisor, substitute *c* for *x*. This gives $c(c-a)(c-b)=17$. The product is positive, so *c* must be positive. This leads to two cases:

I. $0<(c-b)<(c-a)<c$

II. $(c-b)<(c-a)<0<c$

Since 17 is prime, Case I does not occur. In Case II, *c* must be 1 or 17. This gives us $(1-b)<(1-a)<0<1$ and we know that $1(1-a)(1-b)=17$. Thus, $1-a=-1\rightarrow a=2$; $1-b=-17\rightarrow b=18$. $a+b+c=2+18+1=21$. $(17-b)<(17-a)<0<17$ is the other case, so $17-a=-1\rightarrow a=18$; $17-b=-17\rightarrow b=34$. $a+b+c=18+34+17=69$, **C**.

24. Since $\log_4 16=2$, then we must find the values of *x* that generate values of $x-2$ of 1 to 16, inclusive. **A**.

25. Rewrite the problem as $\frac{b}{2}\log\left(\frac{b}{a}\right)-\frac{9}{2}a\log\left(\frac{b}{a}\right)=1$. Let $b=ca$, where *c* is a constant.

Substituting, we get $\frac{ca}{2}\log c-\frac{9}{2}a\log c=1\rightarrow\log c\left(\frac{ca-9a}{2}\right)=1$. This leads us to

$\log c=\frac{2}{a(c-9)}\rightarrow a=\frac{2}{(c-9)\log c}$. Since *a* is an integer, $c=10, a=2, b=20,$ and $b^2-a^2=396$, **B**.

26. Add the second and third equations to get $ac+bd+ad+bc=77\rightarrow(a+b)(c+d)=7\cdot11$ or $1\cdot77$. Since the second product is impossible if the variables are all positive integers, we must have $a+b+c+d=18$. The solution is **D**.

27. For positive numbers *a*, *b*, *c*, *d*, $\frac{a}{b}<\frac{c}{d}$ if and only if $ad<bc$. Here, $3(68)n<17n$ and $51n<68(32)$

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This leaves us with $n>12$ and $n\le 42$, a total of 30 values, **C**.

28. Raising each value to the 30th power gives us $\left(2^{\frac{1}{6}}\right)^{30}=32, \left(3^{\frac{1}{10}}\right)^{30}=27, \left(6^{\frac{1}{15}}\right)^{30}=36$, so **D**.

29. Synthetic division by $x=1$ gives $2x^3+3x^2+8x+12=0$, which can be factored by grouping: $(2x+3)(x^2+4)=0$. $(x-1)(2x+3)$ gives us choice **C**.

30. Complete the square to get the circle $(x-2)^2 \le -\left(y-2\sqrt{3}\right)^2 + 16 \to (x-2)^2 + \left(y-2\sqrt{3}\right)^2 \le 16$. **D**.