2015 Theta Equations and Inequalities
Answers

1. D
2. A
3. B
4. A
5. C
6. B
7. A
8. B
9. D
10. A
11. B
12. D
13. C
14. A
15. B
16. E  Answer: \( \left( \frac{18}{a-6}, \frac{-4(a+3)}{a-6} \right) \)
17. A
18. A
19. B
20. D
21. E  Answer: 2.5
22. C
23. C
24. A
25. B
26. D
27. C
28. D
29. C
30. D
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Solutions

1. To have no real roots, the discriminant must be negative. \( 16k^2 - 4(3k)(1) < 0 \Rightarrow 4k(4k - 3) < 0. \)

   The solution interval is \( \left( 0, \frac{3}{4} \right), D. \)

2. Using Vieta’s formula, the sum is found by \( \frac{(8)}{4} = 2, A. \)

3. \( \log_y x \) is equivalent to \( \frac{1}{\log_x y}. \) After multiplying the entire equation by \( \log_y x \) and

   rearranging, we have \( (\log_y x)^2 - 2.9(\log_y x) + 1 = 0 \Rightarrow 10(\log_y x)^2 - 29(\log_y x) + 10 = 0. \) This factors to

   \( (5\log_y x - 2)(2\log_y x - 5) = 0, \) giving \( \log_y x = \frac{2}{5} \) or \( \frac{5}{2}. \) These are equivalent, respectively, to

   \( x = y^\frac{2}{5} \) or \( y^\frac{5}{2}. \) Substituting these into \( xy = 128, \) we obtain \( y = 32, x = 4 \) and \( y = 4, x = 32. \) Either way, the sum is 36, \( B. \)

4. \( 2^x = \frac{1}{4} = 2^y, \) \( (2)^2 = \frac{1}{4}, A. \)

5. \( 3a^2 - 4ab + b^2 = 0 \Rightarrow (3a - b)(a - b) = 0. \) This gives \( \frac{a}{b} = \frac{1}{3} \) and \( a = b = 1. \) Since \( a \) and \( b \) must be distinct,

   the ratio must be \( \frac{1}{3}, C. \)

6. \( x = 2 \) is the only positive root of the equation. \( x = 2 \) is also a root, along with two imaginary roots.

7. \( x^2 \sqrt{x^2 - 6} = 0 \Rightarrow x^2 \left| x \right| \left| 3 \left( \left| x \right| + 2 \right) \right| = 0. \) The only real solutions come from

   \( \left| x \right| = 3 \Rightarrow x = \pm3. \) These values are found in the interval in \( A. \)

8. Let \( x \) be the number of 5-cent increases. \( R = (0.85 + 0.05x)(5000 - 200x) = 10x^2 - 80x - 2500. \)

   This function has a maximum at \( x = \frac{80}{2(10)} = 4. \) A 20-cent increase bring the total to $1.05, \( B. \)

9. \( 2x \sqrt{2} < 3 \Rightarrow x < 3 + \frac{\sqrt{2}}{2} \approx 3.707 \) and \( 1 \Rightarrow 4 \Rightarrow x > \frac{5}{2}. \) The answer is \( 3, D. \)

10. \( \binom{n}{P_4} = 20 \left( \binom{n}{4} \right) = \frac{20(6!)}{(n - 4)!} \Rightarrow \frac{n(n - 1)(n - 2)(n - 3)(n - 4)!}{(n - 4)!} = \frac{10(n - 1)(n - 2)(n - 3)!}{(n - 4)!}. \)

   This simplifies to \( n(n - 3) = 10 \Rightarrow n = 5, A. \)

11. Squaring each side, we get \( x^2 + 4x + 4 = x^2 + 4x + 8 \Rightarrow x \to x^2 + 4x - x \sqrt{8} \) and \( x^2 + 4x = x(4x - x \sqrt{8}). \) Dividing by \( x, \) we lose a
solution of 0, which won’t affect our final answer. Squaring again, we obtain \((x+4)^2 = 8\) \(x \to x^2 + 9x + 8 = 0\). This factors to \((x+8)(x+1)=0\), but the only true solution we get is \(x = -1\), B.

12. The third side must be between 4 and 24 (exclusive), so the answer is D.

13. Using the quadratic formula, we get \(x = \frac{-3 \pm \sqrt{3^2 - 4(-2)}}{2} = \frac{-6 \pm 2}{2}\), C.

14. Using expansion of minors: \((x - 4)(-3 - 8) - (x - 2)(-1 - 8) + (-2)(2 - 6) = 28\). This simplifies to \(x = 3\), A.

15. \(\log(x) \cdot \frac{3}{2} = \log x \cdot \frac{3}{2} \Rightarrow c^3 = 16 \Rightarrow c(c+4)(c-4) = 0 \Rightarrow \log x = 0, 4, -4\). So, \(x = 1, x = 10,000,\) or \(x = 1\), B.

16. Using substitution, \(ax + 3(-2x - 4) = 6 \Rightarrow x = \frac{18}{a-6}\). \(y = -2\left(\frac{18}{a-6}\right) - 4 \Rightarrow -\frac{4(a+3)}{a-6}\), E.

17. Using “stars and bars,” \(\binom{10+4}{10 \text{ (or 3)}} = 286\), A.

18. \(x^2 + y^2 = 270 \Rightarrow (x+y)(x-y) = 2 \cdot 3^3 \cdot 5\). \(x+y\) and \(x-y\) must be both even or both odd; however, 270 has only one factor of 2 so its factors are one even and one odd. Therefore there are no integral pair factors, A.

19. \(XY \geq 23 = 1XY\) so \(Y = 7\) since \(3 \cdot Y = 21\). \(X7 \cdot 23 = 1X71\), so \((10X+7) \cdot 23 = 1000 + 100X + 10(7) + 1\). This gives \(X = 7\). \(X+Y = 14\), B.

20. Rewrite the problem as \(3^{4x^3} \cdot 3x^2 = 3^{3x}\). Set the exponents equal to one another and multiply by 3 to clear the fraction. This gives \(12x^3 + 24x^2 = 5x\), which has a solution of 0, which does not affect our answer. Divide by \(x\) and then use the quadratic formula:

\[
x = \frac{-24 \pm \sqrt{576 - 4(12)(-5)}}{24} = \frac{-6 \pm \sqrt{51}}{6}.
\]

The sum of these solutions is -2, D.

21. Split the problem into \(3 \cdot 4x \cdot 1\) and \(3 \cdot 4x\). 11. These result in \(x = \frac{1}{2}\) and \(x = 2\). The length of this interval is 2.5, E.
22. Let \( u = \frac{1}{x} \), \( v = \frac{1}{y} \), \( w = \frac{1}{z} \). Now we have \( \begin{cases} u + v = \frac{1}{3} \\ u + w = \frac{1}{5} \\ v + w = \frac{1}{7} \end{cases} \). Subtracting the first two and adding to the third gives us \( v = \frac{29}{210} \). From the third equation, \( w = \frac{1}{7} - \frac{29}{210} = \frac{1}{10} \). \( z = \frac{1}{v} = \frac{v}{w} = 29 \). C.

23. Since \( c \) has to be a divisor, substitute \( c \) for \( x \). This gives \( c(c-b)(c-a)=17 \). The product is positive, so \( c \) must be positive. This leads to two cases:
   I. \( 0<(c-b)<(c-a)<c \)
   II. \( (c-b)<(c-a)<0<c \)

Since 17 is prime, Case I does not occur. In Case II, \( c \) must be 1 or 17. This gives us \( 1-b=18 \Rightarrow b=17 \), \( a+b+c=2+18+1=21 \). \( (17-b)<(17-a)<0<17 \) is the other case, so \( 17-a=-1 \Rightarrow a=18 \); \( 17-b=-17 \Rightarrow b=34 \). \( a+b+c=18+34+17=69 \). C.

24. Since \( \log_4 16=2 \), then we must find the values of \( x \) that generate values of \( \frac{x}{2} \) of 1 to 16, inclusive. A.

25. Rewrite the problem as \( \frac{b}{2} \log \left( \frac{b}{a} \right) = \frac{9}{2} \log \left( \frac{b}{a} \right) = 1 \). Let \( b=ca \), where \( c \) is a constant.

Substituting, we get \( \frac{ca}{2} \log c = 1 \Rightarrow \log c = 1 \Rightarrow c = \frac{9}{2} \). This leads us to \( \log c = \frac{2}{a(c-9)} \Rightarrow c = \frac{2}{(c-9) \log c} \). Since \( a \) is an integer, \( c=10, a=2, b=20 \), and \( b^2 = 396 \). B.

26. Add the second and third equations to get \( ac+bd+ad+bc=77 \Rightarrow (a+b)(c+d)=7 \cdot 11 \) or \( 1 \cdot 77 \).

Since the second product is impossible if the variables are all positive integers, we must have \( a+b+c+d=18 \). The solution is D.

27. For positive numbers \( a, b, c, d \), \( \frac{a}{b} < \frac{c}{d} \) if and only if \( ad < bc \). Here, \( 3(68)n < 17n \) and \( 51n < 68(32) \).

This leaves us with \( n > 12 \) and \( n = 42 \), a total of 30 values. C.

28. Raising each value to the 30th power gives us \( \left( \frac{2}{3} \right)^{30} = 32, \left( \frac{3}{2} \right)^{30} = 27, \left( \frac{6}{5} \right)^{30} = 36 \). So D.

29. Synthetic division by \( x=1 \) gives \( 2x^3 + 3x^2 + 8x + 12 = 0 \), which can be factored by grouping: \( (2x+3)(x^2+4) = 0 \). \( (x-1)(2x+3) \) gives us choice C.
30. Complete the square to get the circle \((x - 2)^2 \leq \left(y - 2\sqrt{3}\right)^2 + 16 \rightarrow (x - 2)^2 + \left(y - 2\sqrt{3}\right)^2 \leq 16\). D.