2015 Theta Geometry Topic Test Answer Key

1. A
2. D
3. C
4. D
5. A
6. B
7. B
8. A
9. B
10. C
11. D
12. D
13. A
14. B
15. A
16. A
17. B
18. E (9)
19. A
20. D
21. A
22. C
23. C
24. D
25. C
26. B
27. C
28. D
29. B
30. A

1. There are 8 triangles to make the full 360° rotation. The smallest has area ½ and the area doubles with each consecutive triangle. You can use the geometric series formula to find the sum, S: $S = \frac{A(1-r^n)}{1-r} = \frac{\frac{1}{2}(1-2^8)}{1-2} = \frac{-\frac{255}{2}}{-1} = \frac{255}{2}$. You could also just add up the areas of the 8 traingles, and that isn't cumbersome at all.

2. Consider all of the Pythagorean Triples with perimeter less than 70.

3-4-5 has perimeter 12. We can multiply this by any number 1 thru 5 (5 triangles)
5-12-13 has P = 30. We can multiply this by 1 or 2 (2 triangles)
7-24-25 has P = 56. We just get 1 triangle here.
9-40-41 is too big!
8-15-17 has P = 40, so we get 1 more.
20-21-29 has P=70, so it just makes it!
28-45-53 is too big!
Then 5 + 2 + 1 + 1 + 1 = 10 possible triangles.

3. In the east-west direction, there is a 100-23 or 77 mile change. In the north-south direction, there is a 300-36 or 264 mile change. This gives us a right triangle with legs 77 and 264, and the hypotenuse is the distance we are looking for. This is a 7-24-25 triangle multiplied by 11, so 11(25) = 275. No need to round.

4. One line goes through (0,-3) and (3,6), and has a slope of 3. The line perpendicular to this has slope $-\frac{1}{3}$, and passes through (3,6) as well. Its equation is $y-6=-\frac{1}{3}(x-3)$ and if we let x=0, we get y=7. So the distance between y-intercepts of 7 and -3 is 10! (Well, just 10.)

5. The surface area of a sphere is $4\pi r^2$. When he cuts it in half, he retains half, or $2\pi r^2$, of the outside of the sphere, but he also adds a circle, or πr^2, to his half. He now has $3\pi r^2$ for the surface area of the fruit in his hand. From 4 circles to 3 circles is a 25% reduction.

6. The complement of an angle x is 90-x. The supplement of an angle x is 180-x. For any angle, the supplement minus the complement becomes $(180-x)-(90-x)=90$.

7. The area is the sum of its non-overlapping parts. Let the vertical columns making the letters go all the way to the top (3 on the M, 2 on the A, and 2 on the O), giving 7 columns with area 2(10) or 20, for a total of 140. The M has 2 additional 2 by 2 squares for a total of 8, and both the A and the O have 2 additional 2 by 4 rectangles for a total of 32 (combined). 140+8+32=180. You may have found an easier way, like subtracting.

8. The number of total paths from A to E passing through C is the product of the number of paths from A to C (which we will call "x") and the number of paths from C to E (which we will call "2x-4"). Then $x(2x-4)=48$, and solving this gives x=6. There are 6 paths from A to C and there are 8 paths from C to E, so E to C has 2 more paths.

9. Consider ¼ of the circle (like the positive x-axis and Quadrant 1). There are 15 points in Quadrant I: (1,1); (1,2); (1,3); (1,4); (2,1); (2,2); (2,3); (2,4); (3,1); (3,2); (3,3); (3,4); (4,1); (4,2); and (4,3). There are 5 on the positive x-axis: (1,0); (2,0); (3,0); (4,0); and (5,0). This gives 20 points in this quarter of the circle. So 20(4)=80, and don't forget the origin, so 81 points.

10. Since the circumference and area are equal, $2\pi r = \pi r^2$, and solving for r, we get r=2 as the only answer we will entertain. A $72°$ arc is 1/5 of the circle, so 1/5 of the circumference, which we now know to be 4π, is $\frac{4\pi}{5}$.

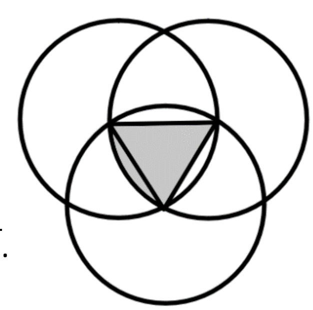
11. Using Power of a Point with respect to A, and letting CE = y we get the equation $15^2 = 9(9+7+y)$. This leads to y = 9. Now, inside the circle, the products of the pieces of these two chords are equal. Thus 7(9) = 6(x), and x = $\frac{21}{2}$.

12. Write an expression for arc ADC as $360 - (x+30)$ or $330 - x$. Then the angle formed by the two tangents is half the difference of the intercepted arcs. So $x = \frac{1}{2}[(330 - x) - (x+30)]$ or $x = \frac{1}{2}[300 - 2x] = 150 - x$, which gives us $x = 75$. As we mentioned the measure of arc ADC is 330-x, which we can find to be 255.

13. Using law of cosines, and letting all of the congruent segments be of length 1, you can find that the side length for triangle ABC is $\sqrt{7}$. This would give a linear ratio of 1: $\sqrt{7}$ and an area ratio of 1:7. Consider ΔAXB with base $\overline{XA}$, and ΔXYZ with base $\overline{XY}$. Their bases are equal, but ΔAXB has twice the height, and thus twice the area. So, if the 3 triangles each have twice the area of the inside equilateral triangle, then the outside equilateral triangle has 7 times the area. Again, the ratio is 1:7.

14. For any point inside an equilateral triangle, the sum of the 3 perpendicular distances to the sides is equal to the height of the triangle. Then the height is 9. Drawing that height, we can see it is the long leg of a 30-60-90 formed by the height, so the short leg is $3\sqrt{3}$, which is half of the base. Area of triangle is half the base times the height, so $3\sqrt{3} \cdot 9 = 27\sqrt{3}$.

15. This problem is sometimes referred to as a British Flag problem. The sums of the squares of distances to point P from opposite vertices are equal. In numbers, $3^2 + 8^2 = 2^2 + TP^2$, and we solve for $TP = \sqrt{69}$.



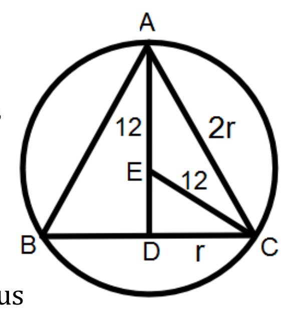
16. Since the circles pass through each other's centers, the shaded triangle Is an equilateral triangle with side length equal to the radius, which is 6. This triangle has area $9\sqrt{3}$. Then the shaded sector is found by subtracting this equilateral triangle from a 60° sector of a circle of radius 6. The area of the sector is $\frac{1}{6}$ of the area of the circle, so $\frac{1}{6} \cdot 36\pi = 6\pi$, minus the area of the triangle, which we found to be $9\sqrt{3}$, giving the sector area $6\pi - 9\sqrt{3}$. Three sectors and a triangle give $3(6\pi - 9\sqrt{3}) + 9\sqrt{3} = 18\pi - 18\sqrt{3}$.

17. If the exterior angles are in an arithmetic progression, then the supplements will be as well. There are 5 angles and they form an arithmetic sequence, so the middle one is actually the average interior angle for a pentagon, 108°. The smallest and largest will actually average to this same middle value, so their sum is 216°.

18. This is a 2-3-4 triangle (not a right triangle), and its perimeter is 9. Notice you cannot use 1 as a side length if the sides have to be DISTINCT integers, as this would fail to satisfy the Triangle Inequality Theorem.

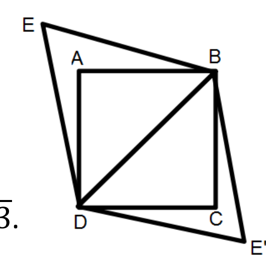
19. Let the distance from the center to the point R be *x*, and we know the radius is 2. Then AR=x-2 and BR = x+2. The sum of these two distances is 2x. Use the distance formula to get PR = $\sqrt{41}$, so $2x = 2\sqrt{41}$.

20. Use the Angle Bisector Theorem to get FR = 12. Then you can see that ΔFOR is a right triangle. The median drawn to the hypotenuse of a right triangle is half of the length of the hypotenuse, so $3\sqrt{5}$.

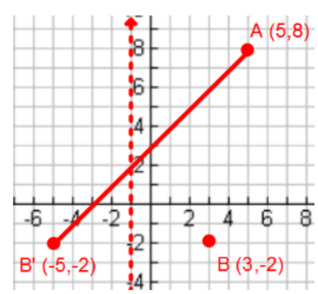


21. For simplicity, a cross section containing the center of the sphere the center of the base of the cone has been included. We know the radius of the sphere is 12, so AE=EC=12. We also know the slant height (AC) of the cone is twice the radius (DC) of the cone. AD is the height of the cone, and we have a right triangle with one leg equal to half the hypotenuse, so it is a 30-60-90, with m∠DAC=30. Isosceles ΔAEC would also have m∠ACE=30. So $m\angle ECD = 30$, and ΔEDC is a 30-60-90. So $ED = 6$ and $DC = r = 6\sqrt{3}$. So the radius of the cone is $6\sqrt{3}$ and the height of the cone (AD) is 18. The volume is then $\frac{1}{3}\pi\left(6\sqrt{3}\right)^2 \cdot 18 = 648\pi$.

22. Let the regular perimeter of each figure be 12. Then the hexagon, with side length 2, is comprised of 6 equilateral triangles, each with area $\sqrt{3}$. The equilateral triangle has side length 4, and area $4\sqrt{3}$. The ratio of the area of the hexagon to the area of the triangle is $6\sqrt{3}:4\sqrt{3}$, which reduced to 3:2.

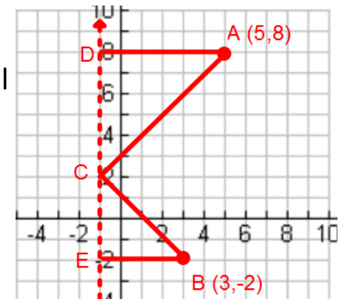


23. The square has area 128, so the side length is $8\sqrt{2}$. The distance between the two possible points for E, will be twice the height of an equilateral triangle with side length equal to 16 (the diagonal of the square). The height of the triangle is $8\sqrt{3}$, so the distance between the two possible points for E is $16\sqrt{3}$.



24. First reflect B over the line x=-1 to get point B'. The shortest route is equal to the segment length from A to B'. (If you want to see the actual path, then reflect the portion of the path to the left of x=-1 over the reflection line.) The length of this path is given by the distance of the segment shown, $10\sqrt{2}$. (The problem would work the same if you reflected point A to get A' instead.)

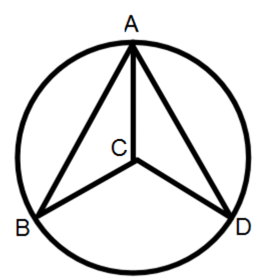
25. When we reflect off of the line x=-1, we create a two congruent angles at the reflection point ($\angle DCA \cong \angle ECB$). Also drawing horizontal segments $\overline{AD}$ and $\overline{EB}$, we now have two similar right triangles, namely ΔADC and ΔBEC. So we know DE = 10, so I will let EC = x and CD = 10-x. Similarity yields the proportion $\frac{AD}{BE} = \frac{DC}{EC}$ or $\frac{6}{4} = \frac{10-x}{x}$. Solving this proportion gives x=4. Move 4 units up from point E to get to point C, whose coordinates are (-1,2).



26. On the sphere and on the cylinder really have no impact in this problem since the volume of "on" these objects is essentially zero. So to figure the probability of Pythagoras being correct, we will write the volume of the space outside the sphere but inside the cylinder over the total volume (which is volume of the cylinder). Now, the sphere and cylinder have the same radius, r, and the height of the cylinder is 2r. The volume of the sphere is $\frac{4}{3}\pi r^3$, and the volume of the cylinder is $\pi r^2 h \text{ or } \pi r^2 (2r) \text{or } 2\pi r^3$. So our probability is given by the expression:
$\frac{2\pi r^3 - \frac{4}{3}\pi r^3}{2\pi r^3} = \frac{\frac{2}{3}\pi r^3}{2\pi r^3} = \frac{1}{3}$.

27. One area formula for a trapezoid is A = mh, and we know the median is 10, and the area is 70. This gives us a height of 7, which is also the diameter of the circle. Now, the drawing is not quite right, as the median will certainly pass through the center of the circle. So EF + GH is basically the length of the median reduced by the diameter of the circle. 10 – 7 = 3.

28. Since $m\angle BCD = 120$, and we have two congruent chords, the other two arcs are congruent, each having a measure of 120. If we then use the $m\angle ACB = 120$, and drop an altitude to the base of isosceles ΔABC, then we bisect the base at point E, and AE=3. We can now find the length of the altitude we dropped to be $3\sqrt{3}$. So the area of ΔACB is $9\sqrt{3}$. Using symmetry, the area of ΔACD Is also $9\sqrt{3}$. This means the area of ABCD is $18\sqrt{3}$.



29. All of the diagonals are only congruent for regular 3-, 4-, and 5-gons. 3+4+5 = 12.

30. First, note that ΔCA_1B_1 is a right triangle. Find the length of the altitude to the hypotenuse to be $\frac{12}{5}$. Since we are extending the sides by 5 and 3 each time, all lines of the form A_nB_n are parallel, and furthermore are separated by a consistent distance of $\frac{12}{5}$. This means that the height of the quadrilateral, which actually is a trapezoid, $A_{100}A_{101}B_{101}B_{100}$ is $\frac{12}{5}$. Now for the bases, it turns out that A_1B_1=5, A_2B_2=10, and so on. In general A_nB_n=5n. So $A_{100}B_{100}$= 500 and $A_{101}B_{101}$= 505. Then the area of the trapezoid is $\frac{1}{2}\left(\frac{12}{5}\right)(500 + 505) = \frac{6}{5}(1005) = 1206$.