Answers:

1. A
2. D
3. B
4. D
5. A
6. A
7. C
8. A
9. C
10. C
11. D
12. C
13. A
14. C
15. B
16. A
17. D
18. D
19. D
20. E
21. A
22. A
23. B
24. C
25. D
26. A
27. D
28. B
29. B
30. C

Solutions:

1. Bringing the power down in front of the last log will allow us to cancel the coefficients of 2 in all terms. Therefore, $\log_3 x = \log_3(1-a) + \log_3(1+a) - \log_3\left(\frac{1-a^2}{a}\right) = \log_3 \frac{(1-a)(1+a)}{\frac{1-a^2}{a}}$

$= \log_3 a \Rightarrow x = a$.

2. $\left(x+y\sqrt{2}\right)^2 = \left(x^2+2y^2\right) + 2xy\sqrt{2}$, so based on the answer choices, $xy = 6$. There are only 4 pairs of positive integers that satisfy this equation: $(1,6)$, $(2,3)$, $(3,2)$, and $(6,1)$. The integer part of the square for each of these is 73, 22, 17, and 38, respectively, so choice D is not possible.

3. $\dfrac{\log_3 \sqrt{243\sqrt{81\sqrt[3]{3}}}}{\log_2 \sqrt[4]{64} + \ln e^{-13}} = \dfrac{43/12}{-23/2} = -\dfrac{43}{138}$

4. $5^{x(x+1)} = 5^{2+4+6+\ldots+2x} = 0.04^{-28} = 5^{56} \Rightarrow x = 7$ (since $x > 0$)

5. $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36} = \log_{36} 2 + \log_{36} 3 = \log_{36} 6 = -\dfrac{1}{2}$

6. $\log f + \log g = \log(1+x+xy+y) = \log(1+x)(1+y)$, and since *f* was a function of *x* only, $f = 1+x$.

7. $(ax)^{\log a} = (bx)^{\log b} \Rightarrow a^{\log(ax)} = b^{\log(bx)} = b^{\log(ax)} b^{\log\frac{b}{a}} \Rightarrow \left(\dfrac{a}{b}\right)^{\log(ax)} = b^{\log\frac{b}{a}} = \left(\dfrac{b}{a}\right)^{\log b} = \left(\dfrac{a}{b}\right)^{-\log b}$

$\Rightarrow \log(ax) = -\log b = \log\dfrac{1}{b} \Rightarrow ax = \dfrac{1}{b} \Rightarrow x = \dfrac{1}{ab}$

8. $10\cdot 5^x = 2\cdot 5^{x+1} = 1 + \dfrac{3}{5^x} \Rightarrow 0 = 10\left(5^x\right)^2 - 5^x - 3 = \left(5\cdot 5^x - 3\right)\left(2\cdot 5^x + 1\right) \Rightarrow 5^x = \dfrac{3}{5}$ (since $5^x > 0$)

$\Rightarrow x = \log_5 \dfrac{3}{5} = \log_5 3 - 1$

9. $(\log_3 p)^2 = \log_3(p^2) = 2\log_3 p \Rightarrow \log_3 p = 0 \text{ or } 2 \Rightarrow p = 1 \text{ or } 9$. Plugging $p = 1$ into the second equation yields $\log_3(1+q) = \log_3 1 + \log_3 q = \log_3 q$, which is impossible. Plugging $p = 9$ into the second equation yields $\log_3(9+q) = \log_3 9 + \log_3 q = \log_3(9q) \Rightarrow 9 + q = 9q \Rightarrow q = \frac{9}{8}$. Therefore, $\frac{q}{p} = \frac{9/8}{9} = \frac{1}{8}$.

10. Each answer choice, when expanded consists of a leading 1, 2315 0s, and the added digit as the last digit. Since the divisibility rule for 9 is that the sum of the digits must be a multiple of 9, the final digit must be one less than a multiple of 9, making C the answer.

11. $(5^n)(12^n)(15^n) = (5 \cdot 12 \cdot 15)^n = 900^n = 30^{2n}$

12. $x^2yz^3 = 7^4 \Rightarrow x^2y = \frac{7^4}{z^3}$. Multiplying this new equation together with the other given one yields $x^3y^3 = x^2y \cdot xy^2 = \frac{7^4}{z^3} \cdot 7^5 = \frac{7^9}{z^3} \Rightarrow (xyz)^3 = x^3y^3z^3 = 7^9 \Rightarrow xyz = 7^3$.

13. $x = \log_{16} 2^{\log_4 8} = \log_4 8 \cdot \log_{16} 2 = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$

14. $\left(\frac{x^{4a-3b-c}}{x^{3a-b}}\right) x^{3c+2b-a} = x^{4a-3b-c-3a+b+3c+2b-a} = x^{2c}$

15. Squaring both sides of the equation yields $4\sqrt{2} = \sqrt{x^2+2} + x + \sqrt{x^2+2} - x - 2\sqrt{2} \Rightarrow 6\sqrt{2} = 2\sqrt{x^2+2} \Rightarrow 18 = x^2 + 2 \Rightarrow x = \pm 4$. $x = 4$ does work, but $x = -4$ does not since the left hand side of the equation in the problem would be negative. Therefore, there is only 1 rational solution.

16. $128 = \frac{4^{3x}}{2^{x+y}} = \frac{2^{6x}}{2^{x+y}} = 2^{5x-y} \Rightarrow 5x - y = 7$, and $25 = \frac{5^{6x-y}}{25^{y-14x}} = \frac{5^{6x-y}}{5^{2y-28x}} = 5^{34x-3y} \Rightarrow 34x - 3y = 2$. Solving this new set of equations yields $x = -1$ and $y = -12$, so $xy = 12$.

17. $f(x) = g(x-2) = h(x-2) + 3$, so *f* is *g* shifted two units to the right (choice I) and *h* shifted two units to the right and up three units (choice III).

18. $n=3^{14}\left(\frac{1}{3^{13}}+\frac{1}{3^{10}}+\frac{1}{3^{8}}\right)=3+3^4+3^6=3+81+729=813$

19. Let $a=\frac{1}{5}\log_5 x$, then $5a=\log_5 x \Rightarrow x=5^{5a}\Rightarrow x^2=5^{10a}$. So $f(a)=5^{10a}\Rightarrow f(x)=5^{10x}$.

20. $6x=\frac{\ln 30}{\ln 3}\Rightarrow x=\frac{\ln 30}{6\ln 3}$, so I is true. Further, $\frac{1}{6}\left(1+\frac{\ln 10}{\ln 3}\right)=\frac{1}{6}\left(\frac{\ln 3+\ln 10}{\ln 3}\right)=\frac{1}{6}\left(\frac{\ln 30}{\ln 3}\right)$, so II is true. Another way to solve the equation is $6x=\log_3 30\Rightarrow x=\frac{1}{6}\log_3 30=\log_3\sqrt[6]{30}$, but we can raise the base and argument of the logarithm and maintain the same value, so $x=\log_{3^6}\left(\sqrt[6]{30}\right)^6$ $=\log_{729}30$, so III is true.

21. $\prod_{n=2}^{63}\log_n(n+1)=\prod_{n=2}^{63}\frac{\ln(n+1)}{\ln n}=\frac{\ln 64}{\ln 2}=\log_2 64=6$

22. $7^4=2401$, so 7 raised to any positive multiple of 4 will also end in 01. Since 2016 is a positive multiple of 4, 7^{2016} ends in 01, and the sum of the digits is 1.

23. Squaring both sides of the equation yields $m+n-2\sqrt{mn}=4\left(2-\sqrt{3}\right)=8-4\sqrt{3}=8-2\sqrt{12}$, so $m+n=8$ and $mn=12\Rightarrow m=6$ and $n=2$ (since $m>n$ must be true); hence, $\frac{m}{n}=\frac{6}{2}=3$.

24. A is not exponential, and B and D are exponential, but their bases are both greater than 1, so they represent exponential growth. C is also exponential, and its base is between 0 and 1, so choice C represents exponential decay.

25. If it takes them *x* hours to paint the house, then they will each complete $\frac{x}{r}$ of the house, where *r* is the amount of time it would take each to paint the house separately. Therefore, $1=\frac{x}{\log_2 e}+\frac{x}{\log_5 e^2}=x\left(\ln 2+\frac{1}{2}\ln 5\right)=x\left(\frac{1}{2}\ln 20\right)\Rightarrow x=2\log_{20}e=\log_{20}e^2$.

26. The length of the radius is $\ln\sqrt{a^3}=\frac{3}{2}\ln a$, and the circumference is $\ln\left(b^{2\pi}\right)=2\pi\ln b$. Therefore, $\frac{3}{2}\ln a=\ln b\Rightarrow\frac{3}{2}=\frac{\ln b}{\ln a}=\log_a b$.

27. A one unit increase in *x* multiplies $f(x)$ by 0.75, resulting in a 25% decrease of that value.

28. Basically we can compare $\sqrt{20}$, $\sqrt[3]{30}$, $\sqrt[4]{40}$, $\sqrt[5]{50}$, and $\sqrt[6]{60}$. In order, these numbers are greater than 4, slightly larger than 3, about 2.5, slightly larger than 2, and less than 2, so Honest Abe is producing less wood each day than the day before. Therefore, his second largest output was on Tuesday.

29. 30 hours elapsed, and the amount of lawrencium remaining is $\frac{25}{200}=\frac{1}{8}=\left(\frac{1}{2}\right)^3$ of the original amount. Therefore, the sample went through 3 complete half-lives in that 30 hours, meaning the half-life of lawrencium is 10 hours.

30. Both graphs are continuous, and based on the shapes of basic logarithmic and exponential functions, the graphs cannot intersect more than twice. Further, at $x=\frac{1}{4}$, the *y*-values for the points on the graphs are $y=1$ and $y=\sqrt[4]{2}$, so the second graph is above the first graph. Then at $x=\frac{1}{2}$, the *y*-values for the points on the graphs are $y=2$ and $y=\sqrt{2}$, so the first graph is above the second graph, meaning there was an intersection point between those *x*-values. Finally, at $x=4$, the *y*-values for the points on the graphs are $y=5$ and $y=16$, so the graphs have crossed again. Since the two graphs cannot intersect more than twice, but we have demonstrated at least two intersections, the graphs must intersect exactly twice. By the way, one of the intersection points is easy to find: $(2,4)$. The other is more difficult to find.